EXTREMAL GRAPH THEORY VIA TWO GEOMETRY QUESTIONS

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Abstract

In this short note, we will give the basic results of extremal graph theory motivated by the two geometry problems.

1. Two Geometry Questions

Motivating questions related to geometry:

1. Given n points in the plane ($\mathbb{R}^2$), at most how many of them can be unit distance apart? (Erdős Unit Distance Problem, 1946)
2. Let $n_0 \geq 2$ be given. Find n such that, any n point in $\mathbb{R}^2$, no three of them are collinear, contains n_0 points in *convex position* (all points are need for the convex hull).

2. Notations

$G = (V, E)$: *Graph with vertex set V and edge set E* (simple graph)

K_m : *The complete graph on m vertices*

$K_{s,t}$: *The complete bipartite graph (V_1, V_2, E) where $|V_1| = s, |V_2| = t$*

$K_{s_1,\ldots,s_r}$: *The complete r-partite graph $(V_1, \ldots, V_r, E)$ where $|V_i| = s_i$*

$\deg(v)$: *The degree of the vertex v*

$e(G)$: *The cardinality of edge set E*, where $G = (V, E)$

$R(n, k)$: *Ramsey Number*

$ex(n, H)$: *The extremal number of H*

$\chi(H)$: *The chromatic number of* H

$T_{n,r}$: *Turán Graph*

$e(X,Y)$: *The number of edges between* X *and* Y

$d(X,Y)$: *The density of* X *and* Y

$\mathbb{P}$: *The set of all prime numbers*

$[n]$: *The set of first* n *positive integers*

$\mathbb{Z}_{\geq 0}$: The set of non-negative integers

3. Ramsey Numbers

$R(n,k)$: The minimal number m such that for every 2-coloring of K_m $\left(K_m : \binom{[m]}{2} = \{A \subseteq [m] : |A| = 2\}\right)$ there exists a blue $K_n \subseteq K_m$ or there exists a red $K_k \subseteq K_m$ i.e. there exists a subset $X \subseteq [m]$ with n elements and $\binom{X}{2}$ (subsets of X of size 2) is colored by blue, or there exists a subset $Y \subseteq [m]$ with k elements and $\binom{Y}{2}$ is colored by red.

- Observe that $R(n,2) = n,\ R(2,k) = k$.
- For $n, k > 2$, one can show that

$$R(n,k) \leq R(n-1,k) + R(n,k-1).$$

By induction, this proves the existence of the Ramsey numbers.

- Graph: 2-graph
  Hypergraph: s-graph (an edge is a set containing s elements!)
- $R_s(n,k),\ (s \geq 2)$: For s-graphs, we color subsets of size s.
- We have the following bound $R_s(n,k) \leq 1 + R_{s-1}(R_s(n-1,k), R_s(n,k-1))$.
  Also, $R_s(n,s) = n$.
- Instead of 2 colors, we can use finitely many colors:

$$R(n_1,\ldots,n_k) \leq R(n_1,\ldots,n_{k-2}, R(n_{k-1},n_k)),$$

(existence of K_{n_i} of color i for some i). Similarly, one has

$$R_s(n_1, \ldots, n_k) \leq R_s(n_1, \ldots, n_{k-2}, R_s(n_{k-1}, n_k)).$$

This proves the existence of the *Ramsey numbers* (multi versions).

4. Forbidding a Subgraph

Theorem 1. *(Mantel, 1907) Every graph on n vertices with edge density greater than $\frac{1}{2} \cdot \frac{n}{n-1}$ contains a triangle.*

Theorem 2. *(Roth, 1953) Every subset of $\mathbb{Z}_{\geq 0}$ with positive upper density contains a 3-term arithmetic progression (3-AP).*

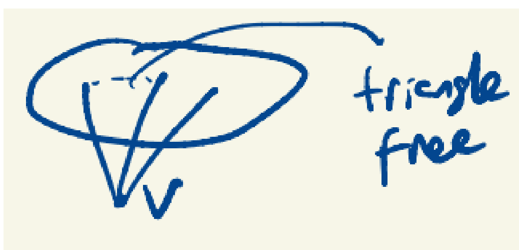
We also note the following two far-reaching results of Szemerédi and Green-Tao.

- Szemerédi (1975): If $A \subseteq \mathbb{Z}_{\geq 0}$ and $\bar{d}(A) > 0$, then A contains a k-AP for every $k \geq 3$.
- Green-Tao (2005): $\mathbb{P}$ contains arbitrarily long APs.

Mantel's theorem is equivalent of the following:

Theorem 3. *(Mantel) Any triangle-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.*

Proof. Let $A \subseteq V$ be a maximum independent set (no edges in A). Observe that the neighborhood of any vertex v is independent by triangle-freeness:



So, $\deg(v) \leq |A|$. Set $B = V \setminus A$. Then, every edge must have a vertex in B. Therefore,

$$e(G) \leq \sum_{v \in B} \deg(v) \leq |A||B| \leq \left(\frac{|A| + |B|}{2}\right)^2 = \frac{n^2}{4}.$$

□

Remark 1. For equality, we need $|A| = |B|$, no edge in B as well, $\deg(v) = |A|$ for all $v \in B$:

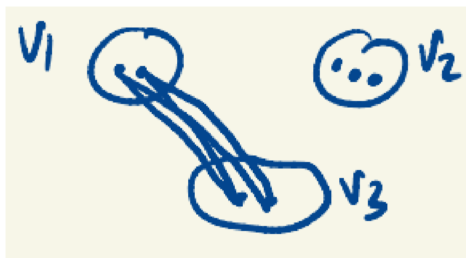
$$K_{\frac{n}{2}, \frac{n}{2}} \text{ for } n \text{ even or } K_{\frac{n-1}{2}, \frac{n+1}{2}} \text{ for } n \text{ odd.}$$

Turán Graph:

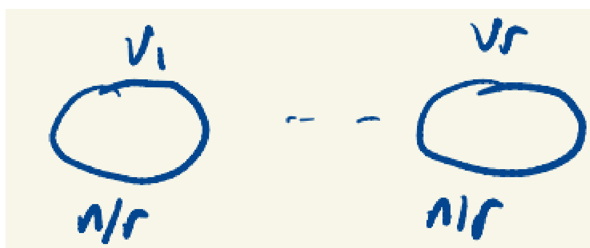
$T_{n,r}$ is the complete r-partite graph $K_{s_1,\dots s_r}$ with

$$n = s_1 + \cdots + s_r \quad \text{and} \quad s_i \in \left\{ \left\lfloor \frac{n}{r} \right\rfloor , \left\lceil \frac{n}{r} \right\rceil \right\}.$$

Example: $T_{8,3} = K_{2,3,3}$ (edges only between V_i, V_j for $i \neq j$)

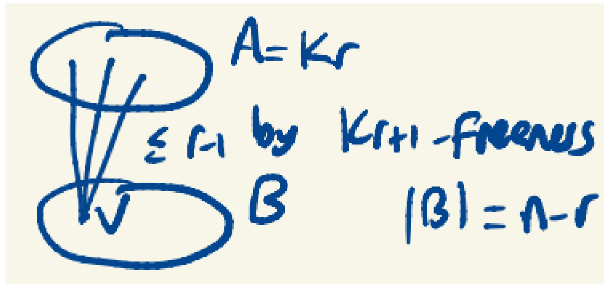


- $e(T_{n,r}) \leq (1 - \frac{1}{r})\frac{n^2}{2} = \binom{r}{2}\frac{n}{r} \cdot \frac{n}{r}$.
  One can easily see this when $r \mid n$.



Theorem 4. *(Turán) Any K_{r+1}-free graph on n vertices has at most $e(T_{n,r})$ edges. ($T_{n,r}$ is the unique maximizer)*

Proof. We proceed by induction on n. If $n \leq r$, then $T_{n,r} = K_n$, and it is the best option. Suppose $n > r$ and the statement holds for all graphs with less than n vertices. Let G be a K_{r+1}-free graph with n vertices and it has the maximal number of edges. Thus, G has a copy of K_r, otherwise we add an edge. Let A be a set of vertices which form a K_r, and set $B = V \setminus A$.
For every $v \in B$, it has at most $r - 1$ neighbors in A.



Hence,

$$e(G) \le \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r}) = e(T_{n,r}).$$

$\square$

- $ex(n,H)$ = the maximum possible number of edges in a H-free graph on n vertices: *the extremal number of H*
- $\chi(H)$ = *the chromatic number* of H

Examples:

$$\chi(K_{r+1}) = r+1$$
$$\chi(T_{n,r}) = r$$

-
$$ex(n, K_{r+1}) = e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$$

  If $\chi(H) = r+1$, then $T_{n,r}$ is indeed H-free and

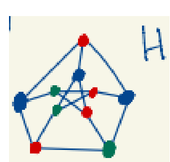
$$ex(n,H) \ge ex(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

  How far can we go? This can be answered by the following result of Erdős-Stone-Simonovits.

Theorem 5. *(Erdős-Stone-Simonovits, 1946)*

$$ex(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}$$

- Petersen graph $= H$, $\chi(H) = 3 = \chi(K_3)$



- There is a proof of ESS theorem which applies Szemerédi's Regularity Lemma. Now, we recall Szemerédi's Regularity Lemma briefly.

For $X, Y \subseteq V$, define

$$e(X,Y) = |\{(x,y) \in X \times Y : xy \in E(G)\}|$$

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

ε-regular pair: Let G be a graph and $U, W \subseteq V$. We call (U,W) an ε-regular pair in G if for all $A \subseteq U, B \subseteq W$ with $|A| \geq \varepsilon |U|, |B| \geq \varepsilon |W|$, one has:

$$|d(U,W) - d(A,B)| \leq \varepsilon.$$

ε-regular partition: The partition $P = \{V_1, \ldots V_k\}, V_1 \sqcup \cdots \sqcup V_k = V$, is an ε-regular partition if

$$\sum_{\substack{(i,j)\in[k]^2 \\ (V_i,V_j) \text{ not } \varepsilon-\text{regular}}} |V_i||V_j| \leq \varepsilon |V|^2.$$

Lemma 1. *(Szemerédi's Regularity Lemma (SGL)) For every $\varepsilon > 0$, there exists $M \in \mathbb{Z}_{\geq 0}$ such that every graph has an ε-regular partition into at most M parts.*

We know that SGL $\implies$ Roth's Theorem, SGL $\implies$ ESS and SGL $\implies$ Szemerédi's Theorem (with more arguments), and hyper-regularity $\implies$ multi-dimensional Szemerédi's Theorem.

- Zarankiewicz Problem: $s \leq t$

Theorem 6. *(Kővári–Sós–Turán, 1954)*

$$ex(n, K_{s,t}) = O_{s,t}(n^{2-\frac{1}{s}})$$

Conjecture 1. (KST)

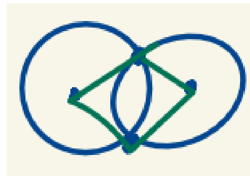
$$ex(n, K_{s,t}) \geq c_{s,t} n^{2-\frac{1}{s}}$$

ESS open for hypergraphs: Even for $K_4^{(3)}$ (bounds via flag algebras)

Back to Erdős unit distance problem: Q1

Given n points in the plane, at most $O(n^{\frac{3}{2}})$ pairs are unit distance apart.

Proof. Consider the graph on those points with edges indicating unit distance. This graph is $K_{2,3}-$free as circles only intersect twice. We are done by KST.



$\square$

Conjecture 2. The true bound for the Erdős unit distance problem is $O_\varepsilon(n^{1+\varepsilon})$.

Theorem 7. *(Erdős-Rényi-Sós, 1966)*

$$ex(n, K_{2,2}) \geq \left(\frac{1}{2} - o(1)\right) n^{\frac{3}{2}}$$

Proof. Let $p = (1 - o(1))\sqrt{n}$ be a prime in $[x - x^{0.525}, x]$ for $x = \sqrt{n}$, by Baker-Harman-Pintz, 2007.
Consider $G = (V, E)$ as follows:

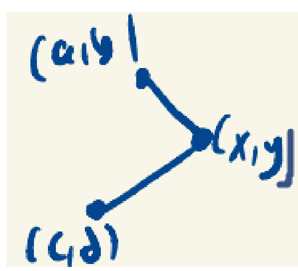
$$\text{for } a, b = 0, 1, \dots, p-1 \quad , \quad (a, b) \neq (0, 0)$$

$$x, y = 0, 1, \dots, p-1 \quad , \quad (x, y) \neq (0, 0)$$

there is an edge between (a, b) and (x, y) if $ax + by \equiv 1 \pmod{p}$. (There are $p^2 - 1$ many such pairs.) For other $n - p^2 + 1$ vertices, they have degree $= 0$, i.e. isolated.
Notice that

$$ax + by \equiv 1 \pmod{p},$$
$$cx + dy \equiv 1 \pmod{p}$$

has at most one solution.



So G is $K_{2,2}-$free, and

$$e(G) = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}(p^2 - 1)(p - 1) = \left(\frac{1}{2} - o(1)\right) n^{\frac{3}{2}}.$$

(a line $ax + by = 1$ has at least $p - 1$ elements) $\square$

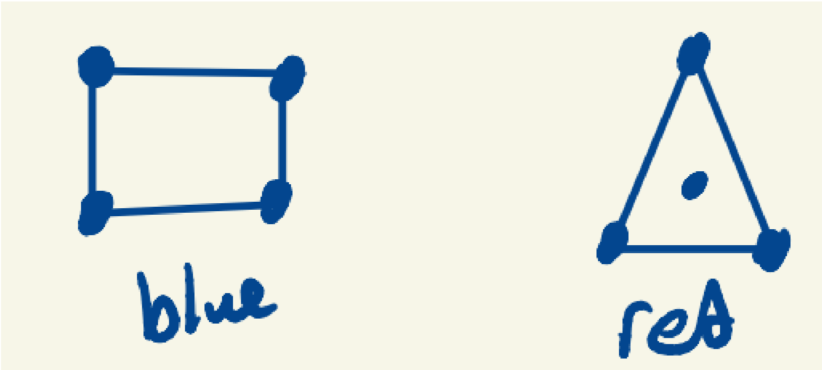
- Brown (1966): $ex(n, K_{3,3}) \geq \left(\frac{1}{2} - o(1)\right) n^{\frac{5}{3}}$

- KST for $K_{4,4}$ open!

Back to Q2:

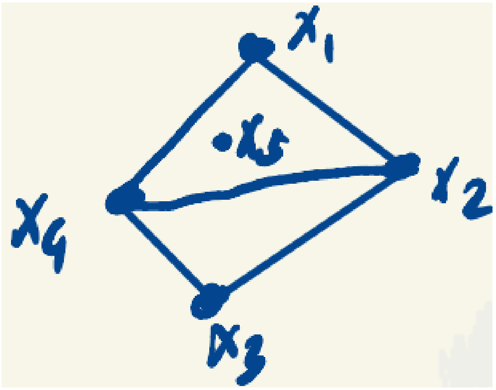
Theorem 8. *(Erdős-Szekeres, 1935) Let $n_0 \geq 2$. There exists a positive integer n such that: Any set of n points in $\mathbb{R}^2$, no three of which are collinear, contains n_0 points in convex position.*

Proof. Let $n = R_4(n_0, 5)$. We give a 4-subset color blue if they are in convex position, otherwise the color is red.



Ramsey theorem ensures the existence of an n_0-subset $S \subseteq [n]$ with all 4-subsets are colored by blue, or a 5-subset with all 4-subsets are colored by red.

First Case: This S is in convex position. Suppose some points are not needed and consider a minimal subset which gives our convex hull (a polygon):

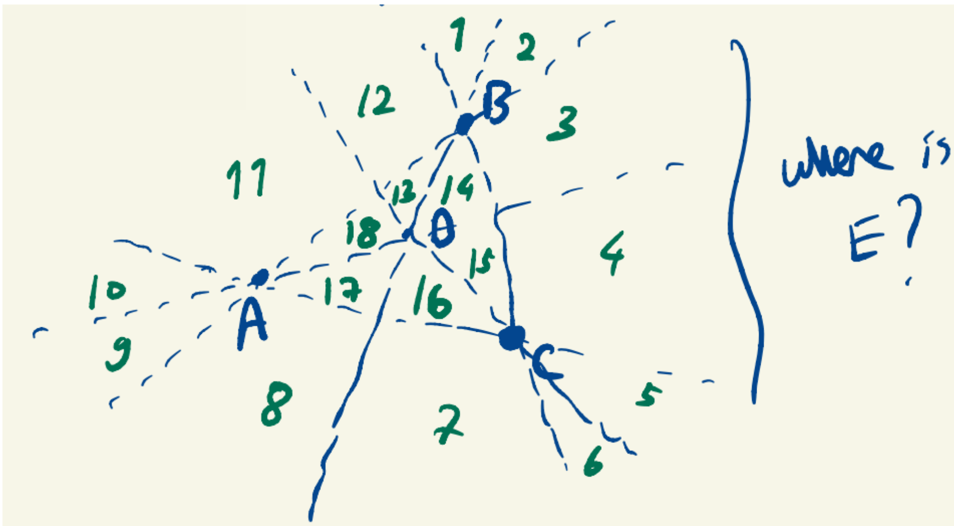


We obtain a 4-subset which is not in convex position. So, we are done.

Second Case: This is impossible due to the *Happy Ending Problem* by Esther. $\square$

Lemma 2. *(Happy Ending Problem) Every set with 5 points in $\mathbb{R}^2$, no three of them lie on a line, always contains 4 points in convex position.*

Proof. The following figure is the proof:



Enough to check: 2,3,4,5 (BCDE), 14 (ABDE), 15 (ACDE) □

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References: Yufei Zhao, Graph theory and additive combinatorics. Cambridge University Press (2021).